# A UNIFIED APPROACH TO THE STUDY OF SUMS, PRODUCTS, TIME-AGGREGATION AND OTHER FUNCTIONS OF ARMA PROCESSES

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**Abstract.** Conditions under which sums, products and time-aggregation of ARMA processes follow ARMA models are derived from a single theorem. This characterizes these processes in terms of difference equations satisfied by their autocovariance function. From this we obtain necessary and sufficient conditions for a function of a Gaussian ARMA process and the product of two possibly dependent Gaussian ARMA processes to be ARMA. We show that the sum and product of two ARMA processes related by a Box and Jenkins transfer function model belong to the ARMA family.

**Keywords.** Autocovariance function; autoregressive-moving average model; Box and Jenkins transfer function model; time series interpretation; transformed variables.

## 1. INTRODUCTION

The family of ARMA processes, popularized towards 1970 by Box and Jenkins (1976), have become an important tool for modelling time series. In fact, the first two moments of any second order stationary, purely non deterministic process may be efficiently approximated by those of a member of the ARMA family, this being equivalent in the frequency domain to approximating a function of $L_1(-\pi, \pi)$ by a rational function in $e^{i\lambda}$. This, in its own right, justifies the study of conditions under which sums, products, time-aggregation and other functions of these processes satisfy an ARMA model.

A second reason for studying this topic is that, especially in certain applications of time series to economics, the families of AR and MA processes are easier to interpret than is the mixed form. Hence it is interesting to show how ARMA processes may arise in practice as sums, products, time-aggregation and other functions of AR and/or MA processes.

Granger and Morris (1976) showed that the sum of uncorrelated ARMA processes is an ARMA process. Introducing an obvious notation we may write

$$\sum_{i=1}^{N} \text{ARMA}(p_i, q_i) = \text{ARMA}(p, q),$$

where

$$p \leqslant \sum_{i=1}^{N} p_i \quad \text{and} \quad q \leqslant p + \max(q_i - p_i, i = 1, \ldots, N).$$

Dossou-Gbete, Ettinger and de Falguerolles (1980) showed that a necessary and sufficient condition for the sum of two possibly dependent ARMA processes

0143-9782/84/030159-13 $02.50/0
JOURNAL OF TIME SERIES ANALYSIS Vol. 5, No. 3

to belong to the same family is that the sum of the corresponding cross-spectral densities be a rational function in $e^{i\lambda}$.

Dossou-Gbete et al. (1980) also proved that the product of independent ARMA processes is ARMA. They did not obtain bounds for the orders of the resulting process.

Finally, for the time-aggregation of ARMA processes, Dossou-Gbete et al. (1980) (based on the work of Amemiya and Wu (1972), Anderson (1976), and Wei (1979)) showed that if $X$ is ARMA $(p, q)$ and $Y$ is defined as

$$Y_t = \sum_{j=0}^{n-1} a_j X_{tl+j}, \tag{1.1}$$

then $Y$ will be ARMA $(p_0, q_0)$ with $p_0 \leq p$ and $q_0 \leq p + [(q-p+n-1)/l]$, where $[x]$ denotes the integer part of $x$.

In the present article we prove these previous results from a single theorem, a characterization of ARMA processes in terms of difference equations satisfied by their autocovariance functions. Though this theorem seems to have first been proved as a lemma in an appendix of Beguin, Gourieroux and Monfort (1980), Professor H. Rost pointed out to me that the ideas for a proof are implicit in work of F. Riesz and L. Fejér dating back to 1916 (cf. Polyá and Szegö (1954, pp. 81–82 and 274–275)). All these proofs are based on frequency domain arguments. We give a simple time domain proof in section 3 which we feel gives more insight into this important result.

Based on this theorem we prove in section 5 that the product of independent ARMA processes is such that

$$\prod_{i=1}^{N} \text{ARMA}\,(p_i, q_i) = \text{ARMA}\,(p, q),$$

where

$$p \leq \prod_{i=1}^{N} p_i \quad \text{and} \quad q \leq p + \max\,(q_i - p_i, i = 1, \ldots, N).$$

We also find a necessary and sufficient condition for the product of two possibly dependent Gaussian ARMA processes to be ARMA. Granger and Newbold (1976) proved that a quadratic transformation of a Gaussian AR $(p)$ process is ARMA. In section 6 we prove that a function of a Gaussian ARMA process will be ARMA if and only if the original process was MA or the function is a polynomial. Finally, we prove that the sum and product of two ARMA processes related by a Box and Jenkins transfer function model belong to the ARMA family.

## 2. DEFINITIONS AND NOTATION

We work with real, zero mean, second order stationary, purely non deterministic processes in discrete time. If $X = (X_t, t \in \mathbb{Z})$ is such a process, $r_X(k)$ will denote its autocovariance function and $H_t(X)$ the Hilbert space spanned by $(X_s, s \leq t)$ with the usual inner product: $\langle X, Y \rangle = E(XY)$.

We say that $X$ is an autoregressive-moving average process of order $(p, q)$ (abbreviated ARMA $(p, q)$) if there exists a zero mean, second order stationary, uncorrelated process $\varepsilon=(\varepsilon_t, t\in\mathbb{Z})$ (called white noise) and polynomials

$$a(z)=1-\sum_{j=1}^{p} a_j z^j, \qquad a_p\neq 0,$$

with roots outside the unit circle and

$$c(z)=1-\sum_{k=1}^{q} c_k z^k, \qquad c_q\neq 0,$$

with roots outside or on the unit circle, such that

$$X_t-a_1X_{t-1}-\cdots-a_pX_{t-p}=\varepsilon_t-c_1\varepsilon_{t-1}-\cdots-c_q\varepsilon_{t-q},$$

or equivalently, denoting by $B$ the backshift operator,

$$a(B)X_t=c(B)\varepsilon_t. \tag{2.1}$$

The operator inverse to $B$ is denoted by $F$.

From the hypotheses made on the roots of $a(z)$ and $c(z)$ we have

$$H_t(X)=H_t(\varepsilon). \tag{2.2}$$

In fact, given any zero mean, second order stationary, purely non deterministic process $X=(X_t, t\in\mathbb{Z})$ (in particular a process $X$ satisfying (2.1)) there exists a white noise process $\varepsilon=(\varepsilon_t, t\in\mathbb{Z})$ and a square summable sequence $(c_j)_{j\geqslant 0}$, $c_0=1$, such that

$$X_t=\sum_{j=0}^{+\infty} c_j\varepsilon_{t-j}, \tag{2.3}$$

where $X$ and $\varepsilon$ satisfy (2.2).

Multiplying both sides of (2.1) by $X_{t-k}$, taking expectations and using (2.2) and (2.3) we obtain that

$$a(B)r_x(k)=0, \qquad k>q \tag{2.4}$$

$$a(B)r_x(q)=-c_q\sigma^2\neq 0, \tag{2.5}$$

where $\sigma^2$ is the variance of $\varepsilon_t$, which we assume positive.

Solving (2.4) we obtain that

$$r_x(k)=\sum_{i=1}^{p} c_i\alpha_i^k, \qquad k>q-p, \tag{2.6}$$

where $\alpha_i^{-1}$, $i=1,\ldots,p$, are the roots of $a(z)$ which, merely to simplify the algebraic expressions involved, we have assumed are distinct.

A similar expression to (2.6) may be obtained for the autocorrelation function of the process, defined by:

$$\rho_x(k)=\frac{r_x(k)}{r_x(0)}.$$

We say that $X$ and $Y$ are related by a Box and Jenkins transfer function model if they are jointly second-order stationary ARMA processes and there exist polynomials

$$\delta(z)=1-\sum_{j=1}^{r}\delta_j z^j, \qquad \delta_r\neq 0,$$

with roots outside the unit circle and

$$\omega(z)=\omega_0-\sum_{k=1}^{s}\omega_k z^k, \qquad \omega_s\neq 0,$$

such that

$$\delta(B)Y_t=\omega(B)X_t+\delta(B)N_t, \tag{2.7}$$

where $N$ is an ARMA process uncorrelated with $X$.

Multiplying both members of (2.7) by $X_{t-k}$ and taking expected value, we obtain

$$\delta(B)r_{xy}(k)=\omega(B)r_x(k), \tag{2.8}$$

where $r_{xy}(k)$ denotes the cross-covariance function of $X$ and $Y$.

Replacing $k$ by $-k$ in (2.8) and using the fact that $r_x(k)$ is even, we have

$$\delta(F)r_{yx}(k)=\omega(F)r_x(k). \tag{2.9}$$

From (2.4), (2.8) and (2.9) it follows that

$$a(B)\delta(B)r_{xy}(k)=0, \qquad k>q+s \tag{2.10}$$

$$a(B)\delta(F)r_{yx}(k)=0, \qquad k>q \tag{2.11}$$

As $r_{yx}(k)$ is bounded and the roots of $\delta(z)$ are outside the unit circle, (2.11) leads to

$$a(B)r_{yx}(k)=0, \qquad k>q. \tag{2.12}$$

Finally we mention that the cross-spectral density between $X$ and $Y$ is defined for $\lambda\, e[-\pi,\pi]$ as

$$f_{xy}(\lambda)=\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} r_{xy}(k)\, e^{-i\lambda k}. \tag{2.13}$$

## 3. THE MAIN RESULT

THEOREM 1. *A necessary and sufficient condition for a zero mean, second-order stationary, purely non deterministic process $X$ to be* ARMA $(p, q)$ *is that there exists a polynomial of degree $p$, $a(z)$, with roots outside the unit circle, such that*

$$\begin{aligned} a(B)r_x(k)&=0, \qquad k>q \\ a(B)r_x(q)&\neq 0. \end{aligned} \tag{3.1}$$

## 4. SUMS OF ARMA PROCESSES

PROPOSITION 1. *Let X and Y be* ARMA $(p_1, q_1)$ *and* ARMA $(p_2, q_2)$, *respectively. A necessary and sufficient condition that* $X+Y$ *be an* ARMA *process is that there exist a polynomial* $a(z)$ *with roots outside the unit circle, such that*

$$a(B)(r_{xy}(k)+r_{yx}(k))=0, \qquad k>q_3. \tag{4.1}$$

*Then* $X+Y$ *will be an* ARMA $(p, q)$ *with* $p\leq\sum_{j=1}^{3} p_j$ *and* $q\leq p+\max(q_i-p_i, i=1,2,3)$ *where* $p_3$ *is the degree of* $a(z)$.

PROOF. Noting that

$$r_{x+y}(k)=r_x(k)+r_{xy}(k)+r_{yx}(k)+r_y(k) \tag{4.2}$$

we have, from (2.4) and (4.1), that

$$a_x(B)a_y(B)a(B)r_{x+y}(k)=0, \qquad k>\sum_{j=1}^{3} p_j+\max(q_i-p_i, i=1,2,3)$$

where $a_x(z)$ and $a_y(z)$ are the autoregressive polynomials of $X$ and $Y$, respectively. The proof concludes by applying theorem 1 to this expression.

REMARKS. 1. If $X$ and $Y$ are uncorrelated then $p_3=q_3=0$ and we recover the result due to Granger and Morris (1976) stated in the introduction.

2. Based on (2.13) and some elementary difference equation theory it is easy to prove that (4.1) implies that $f_{xy}(\lambda)+f_{yx}(\lambda)$ is a rational function in $e^{i\lambda}$. Thus Proposition 1 is a time domain version of the theorem due to Dossou-Gbete et al. (1980).

COROLLARY 1. *If the processes considered in proposition* 1 *are related by a Box and Jenkins transfer function model (see* (2.7)) *then due to* (2.10) *and* (2.12) *we have that*

$$a_x(B)\delta(B)(r_{xy}(k)+r_{yx}(k))=0, \qquad k>q_1+\max(r, s).$$

*Hence, by applying proposition* 1 *(with* $p_3=p_1+r$ *and* $q_3=q_1+\max(r, s)$*) we conclude that* $X+Y$ *is* ARMA $(p, q)$ *with* $p\leq 2p_1+p_2+r$ *and* $q\leq p+\max(q_1-p_1, q_2-p_2, (q_1-p_1)+(s-r))$.

## 5. PRODUCTS OF ARMA PROCESSES

PROPOSITION 2. *Let X and Y be independent* ARMA *processes of order* $(p_1, q_1)$ *and* $(p_2, q_2)$ *respectively and let Z denote their product. Then Z will be* ARMA $(p, q)$ *with* $p\leq p_1p_2$ *and* $q\leq p+\max(q_1-p_1, q_2-p_2)$.

PROOF. Since $X$ and $Y$ are independent it is easy to show that $Z$ will be zero mean, second-order stationary, with

$$r_z(k)=r_x(k)r_y(k). \tag{5.1}$$

By (2.6) we have that $r_x(k)=\sum_{i=1}^{p_1} c_{1i}\alpha_{1i}^k$, $k>q_1-p_1$, and $r_y(k)=\sum_{j=1}^{p_2} c_{2j}\alpha_{2j}^k$, $k>q_2-p_2$. Hence

$$r_z(k)=\sum_{i=1}^{p_1}\sum_{j=1}^{p_2} c_{1i}c_{2j}(\alpha_{1i}\alpha_{2j})^k, \qquad k>\max(q_1-p_1, q_2-p_2).$$

Defining

$$a(z)=\prod_{i=1}^{p_1}\prod_{j=1}^{p_2}(1-\alpha_{1i}\alpha_{2j}z), \tag{5.2}$$

we have that

$$a(B)r_z(k)=0, \qquad k>p_1p_2+\max(q_1-p_1, q_2-p_2).$$

Applying theorem 1 concludes the proof.

Remarks. 1. From (5.2) the roots of $a(z)$ will be of the form $\alpha_{1i}^{-1}\alpha_{2j}^{-1}$, that is, the product of the roots of the autoregressive polynomials of $X$ and $Y$. Hence it is not necessary that $a(z)$ and $c(z)$ have common roots to have $p<p_1p_2$. This will be the case if $\alpha_{1i}\alpha_{2j}=\alpha_{1m}\alpha_{2n}$ for some $i\neq m$, $j\neq n$.

2. Introducing the obvious notation (where strictly speaking the orders on the right-hand side are only upper bounds) we have that for products of independent processes:

$$\mathrm{AR}(p_1)\mathrm{AR}(p_2)=\mathrm{ARMA}\{p_1p_2, p_1p_2-\min(p_1, p_2)\}$$
$$\mathrm{AR}(p)\mathrm{AR}(p)=\mathrm{ARMA}(p^2, p^2-p).$$

On the other hand, as a consequence of (5.1):

$$\mathrm{ARMA}(p_1, q_1)\,\mathrm{MA}(q_2)=\mathrm{MA}(q_2).$$

Thus ARMA processes may arise as the product of two independent AR processes. One situation where an economic time series may be interpreted this way is when $Z$ is the demand for a product satisfied by a certain company, $X$ the nationwide demand for the product and $Y$ the company's market share. That $X$ and $Y$ are independent may be expected to occur frequently. Other examples are mentioned by Wecker (1978).

3. Proposition 2 is easily generalized to the product of $N$ independent ARMA processes:

$$\prod_{i=1}^{N}\mathrm{ARMA}(p_i, q_i)=\mathrm{ARMA}(p, q),$$

where $p\leq\prod_{i=1}^{N} p_i$ and $q\leq p+\max(q_i-p_i, i=1, \ldots, N)$. Note the similarity to the analogous result for sums of uncorrelated ARMA processes.

Proposition 3. *Let $X$ and $Y$ be two possibly dependent Gaussian ARMA processes of order $(p_1, q_1)$ and $(p_2, q_2)$ respectively and let $Z$ denote their product.*

*A necessary and sufficient condition for $Z$ to be* ARMA (*after subtracting its mean*) *is that there exist a polynomial $d(z)$ with roots outside the unit circle, such that*

$$d(B)r_{xy}(k)r_{yx}(k)=0, \qquad k>q_3.$$

*Further, if the degree of $d(z)$ is $p_3$, then $X$ will be* ARMA $(p, q)$ *with $p \leqslant p_1p_2+p_3$ and $q \leqslant p+\max(q_i-p_i, i=1,2,3)$.*

PROOF. We have (Isserlis, 1918) that

$$r_z(k)=r_x(k)r_y(k)+r_{xy}(k)r_{yx}(k). \tag{5.3}$$

Adopting the notation used when proving proposition 2, we shall have, from (5.3),

$$d(B)a(B)r_z(k)=0, \qquad k>q'$$

where $q'=p_1p_2+p_3+\max(q_i-p_i, i=1,2,3)$ *and hence, by applying theorem* 1, *the proof concludes.*

COROLLARY 1. *If the processes considered in proposition* 3 *are related by a Box and Jenkins transfer function model (see* (2.7)) *then, due to* (2.10) *and* (2.12), *$r_{xy}(k)\cdot r_{yx}(k)$ will be of the form*

$$r_{xy}(k)r_{yx}(k)=\sum_{i=1}^{p_1+r}\sum_{j=1}^{p_1}c_{1i}c_{2j}(\alpha_i\alpha_j)^k, \qquad k>q_1-p_1+\max(0, s-r).$$

*By applying proposition* 3 *we conclude that $Z$ is* ARMA$(p,q)$ *with $p \leqslant p_1p_2+\frac{1}{2}(p_1+1)(p_1+2r)$ and $q \leqslant p+\max(q_1-p_1, q_2-p_2, (q_1-p_1)+(s-r))$.*

REMARK. Based on the expression for the autocovariance of the product of two stationary processes due to Wecker (1978), it is easy to state a necessary and sufficient condition for the product of two not necessarily Gaussian ARMA processes to be ARMA.

## 6. FUNCTIONS OF AN ARMA PROCESS

THEOREM 2. *Let $T(x)$ have a Hermite polynomial expansion and $X$ be a Gaussian* ARMA *process. Then a necessary and sufficient condition for $T(X)$ to be* ARMA *is that $X$ be* MA *or $T$ be a polynomial.*

PROOF. From theorem 1 it follows easily that a function of a Gaussian MA $(q)$ process will be MA $(q')$, $q' \leqslant q$. Hence we restrict the proof to the case in which $X$ is ARMA $(p, q)$ with $p \geqslant 1$.

To prove sufficiency, we note that due to Isserlis (1918) for a zero mean Gaussian process $X$

$$E(X_{t_1}X_{t_2}\cdots X_{t_n})=\sum E(X_{t_{i_1}}X_{t_{i_2}})\cdots E(X_{t_{i_{n-1}}}X_{t_{i_n}}), \tag{6.1}$$

*where $n$* is even and the sum is taken over all possible ways of dividing $n$ points into $\frac{1}{2}n$ pairs. If $n$ is odd the corresponding expression is zero. Hence the autocovariance function of a polynomial of degree $n$ in $X$, $Y$, will be a polynomial of degree $n$ in $r_x$:

$$r_y(k)=\sum_{j=1}^{n} c_j\{r_x(k)\}^j, \tag{6.2}$$

where we have omitted the constant term because $r_y(k)$ must tend to zero as $k$ tends to infinity. By substituting (2.6) into (6.2) and applying theorem 1 the desired conclusion follows.

To prove necessity we consider

$$Y_t=T(X_t),$$

where $T$ is a function with Hermite polynomials expansion

$$T(z)=\sum_{j=0}^{+\infty} \alpha_j H_j(z).$$

Based on Granger and Newbold (1976) we have that

$$r_y(k)=\sum_{j=1}^{+\infty} \alpha_j^2 j!\, r_x^j(k), \tag{6.3}$$

where, without loss of generality, we assume $r_x(0)=1$.

If $T$ is not a polynomial, an infinity of $\alpha_j$'s will be non-zero. To prove that $Y$ is not ARMA, due to theorem 1 it suffices to show that in this case $r_y$ cannot be expressed as

$$r_y(k)=\sum_{j=1}^{s} c_j\beta_j^k, \qquad k\geq t. \tag{6.4}$$

As $r_x(k)$ admits an expression like (2.6), by subtracting (6.3) from (6.4) our problem is equivalent to showing that the following situation may not arise:

$$(\forall k\geq s)\sum_{j=0}^{+\infty} k_j\gamma_j^k=0$$

with $\gamma_i\neq\gamma_j$ if $i\neq j$; $0<|\gamma_i|<1$; and infinitely many $k_j$'s different from zero. We may further assume that only a finite number of $\gamma_i$ have modulus equal to

$$k=\max\{|\gamma_i|/i\in IN\},$$

and that they are $\gamma_0, \gamma_1, \dots, \gamma_r$ with $k_i\neq 0$, $i=0,\dots,r$.

Finally, we may take

$$\sum_{j=0}^{+\infty} |k_j|<+\infty.$$

Should this not be the case, due to (6.3), $Y$ would have infinite variance and hence not be ARMA.

That the preceeding situation may not arise, we prove in the Appendix. The basic idea of the proof is due to J. Marhoul.

COROLLARY 1. *In the particular case of the square of a Gaussian* ARMA $(p, q)$ *process we have that* (6.1) *leads to*

$$r_y(k) = 2r_x^2(k).$$

Adopting the notation of (2.6) and defining

$$a(z) = \prod_{i=1}^{p} \prod_{j=i}^{p} (1 - \alpha_i \alpha_j z),$$

it follows that

$$a(B) r_y(k) = 0, \qquad k > \tfrac{1}{2}p(p+1) + q - p,$$

and hence, after subtracting its mean, $X^2$ will be ARMA $(p_0, q_0)$ with $p_0 \leq \frac{1}{2}p(p+1)$ and $q_0 \leq p_0 + q - p$.

Therefore, if $X$ is Gaussian AR $(p)$ then, after subtracting its mean, $X^2$ will generally be ARMA $(\frac{1}{2}p(p+1), \frac{1}{2}p(p-1))$. Of course ARMA processes may also arise as the square of a simpler AR process. For example, if the potential difference through a capacitor follows an AR $(p)$ process, the accumulated charge will be ARMA $(\frac{1}{2}p(p+1), \frac{1}{2}p(p-1))$. Similarly, the power dissipated by a resistance will be ARMA if the corresponding current intensity is AR.

## 7. TIME-AGGREGATION OF ARMA PROCESS

PROPOSITION 4. *Let $X$ be* ARMA $(p, q)$ *and define $Y$ as in* (1.1). *$Y_t$ is a linear combination of $n$ variables of the original process, ranging from $tl$ to $tl + n - 1$. Then $Y$ will be* ARMA $(p_0, q_0)$ *with $p_0 \leq p$ and $q_0 \leq [(q - p + n - 1)/l]$.*

PROOF. It is easy to show that $Y$ will be zero mean, second order stationary, with

$$r_y(k) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_i a_j r_x(kl + j - i). \tag{7.1}$$

The proof then follows by substituting (2.6) into (7.1) and applying theorem 1. The corresponding autoregressive polynomial will be

$$a(z) = \prod_{i=1}^{p} (1 - \alpha_i^l z). \tag{7.2}$$

REMARKS. 1. Both the processes obtained by sampling $X$ at equal intervals in time and by aggregating it in time are particular cases of proposition 4.

2. Under the hypothesis of proposition 4, if $n$ is fixed and $l$ sufficiently large, the process will be ARMA $(p', q')$ with $p' \leq p$ and $q' \leq p$ if $q - p + n - 1 < 0$ and $q' \leq p + 1$ otherwise. Due to (7.1), the limiting process as $l$ tends to infinity will be white noise.

3. A particular case of interest in proposition 4 is when $Y_t$ corresponds to non-overlapping averages of length $n$ of the original process. In this case $n = l$

and $a_j = 1/n, j = 1, \ldots, n$. Cox (1981) notes that $Y$ converges to white noise as $n$ tends to infinity. We may now see how the white noise process is approached.

If we write

$$Y_t^{(n)} = \frac{1}{n} \sum_{j=0}^{n-1} X_{nt+j}$$

we shall have that for sufficiently large $n$, $Y^{(n)}$ will be ARMA $(p, p+1)$ if $p \geq q-1$ and ARMA $(p, p+2)$ otherwise. Because of (7.2) the corresponding autoregressive polynomial will be

$$a_n(z) = \prod_{i=1}^{p} (1 - \alpha_i^n z),$$

thus showing that the roots of $a_n(z)$ will tend to zero geometrically as $n$ increases. A similar situation arises as $l$ tends to infinity for the process considered in the previous remark.

Let us finally mention that ARMA processes may also arise in practice as the result of aggregating or sampling simpler autoregressive processes.

COROLLARY 1. *If any two of the processes $Y$, $X$ and $N$ considered in a Box and Jenkins transfer function model are* ARMA *then the third will also be* ARMA. *Further, if $(p_y, q_y)$, $(p_x, q_x)$ and $(p_N, q_N)$ are the corresponding orders, then $p_y \leq p_x + p_N + r$, $q_y \leq p_N + q_x + s$, $q_y \leq p_x + q_N + r$, $p_x \leq p_N + p_y + s$, $q_x \leq p_y + q_N + r$, $q_x \leq p_N + q_y + r$, $p_N \leq p_x + p_y + r$, $q_N \leq p_x + q_y + r$ and $q_N \leq p_y + q_x + s$.*

PROOF. If $X$ and $N$ are ARMA, the results concerning $Y$ follow from applying proposition 4 to $\omega(B)X_t$ and $\delta(B)N_t$ and then applying proposition 1 to their sum.

If $Y$ and $N$ are ARMA then, noting that the autocovariance function of $Y-N$ is equal to $r_y(k) - r_N(k)$ and applying theorem 1, $Y-N$ will also be ARMA. This part of the proof concludes by noting that

$$\omega(B)X_t = \delta(B)(Y_t - N_t),$$

and applying proposition 4.

Finally, if $X$ and $Y$ are ARMA then, by proposition 4, $\omega(B)X_t$ and $\delta(B)Y_t$ will also be ARMA. Considering that

$$\delta(B)N_t = \delta(B)Y_t - \omega(B)X_t,$$

and noting that the autocovariance function of the left member equals the difference of the autocovariance functions of the right members, we complete the proof by applying theorem 1.

## ACKNOWLEDGEMENTS

I wish to thank Professors D. R. Cox and B. D. Ripley, and the referee for their helpful comments.

## APPENDIX

The following situation may not arise:

$$(\forall k \geq s) \sum_{j=0}^{+\infty} k_j \gamma_j^k = 0 \tag{A.1}$$

$$i \neq j \Rightarrow \gamma_i \neq \gamma_j \tag{A.2}$$

$$k_i \neq 0, \qquad i = 0, \ldots, r \tag{A.3}$$

$$0 < |\gamma_i| = k < 1, \qquad i = 0, \ldots, r \tag{A.4}$$

$$|\gamma_j| < k, \qquad j > r \tag{A.5}$$

$$\sum_{j=0}^{+\infty} |k_j| < +\infty \tag{A.6}$$

where the $k_i$'s and $\gamma_i$'s are complex numbers.

PROOF. From (A.1):

$$(\forall k \geq s) - \sum_{j=0}^{r} k_j \left(\frac{\gamma_j}{\gamma_0}\right)^k = \sum_{j=r+1}^{+\infty} k_j \left(\frac{\gamma_j}{\gamma_0}\right)^k \tag{A.7}$$

By applying Lebesgue's Convergence Theorem to $f_n = f^n$, with $f(j) = |\gamma_j|/|\gamma_0|$ if $j \in T = \{r+1, r+2, \ldots\}$ and 0 elsewhere and taking the measure $\mu$ which associates to $j$ mass $|k_j|, j \in T$, we conclude that

$$\lim_{k \to +\infty} \sum_{j=r+1}^{+\infty} |k_j| \left|\frac{\gamma_j}{\gamma_0}\right|^k = 0. \tag{A.8}$$

(Note: Due to (A.4) and (A.5), $f_n$ will be dominated by the natural number's indicator function which will be integrable due to (A.6).)

But the left side of (A.7) corresponds to the solution of a homogeneous linear difference equation of order $r+1$ with constant coefficients and all its characteristic values on the unit circle. It is well known that such an expression has no limit when $k$ grows, contradicting (A.8).

## NOTE

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