What do "Let's Make a Deal" and "Ask Marilyn" have in common? A probability problem.

Monty Hall's Probability Puzzle

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Introduction

The game played on the TV show "Let's Make a Deal" (with Monty Hall) gives rise to the following interesting puzzle. There are three curtains, an announcer (Monty) and a player. One of the curtains contains a valuable prize and the other two are empty; only Monty knows where the prize is. Players win the prize if they guess correctly which curtain contains the prize. Initially the player selects a curtain. Then, Monty opens an empty curtain that the player did not choose. Having eliminated that curtain from contention, Monty offers the player the option to select the remaining third curtain as a final choice. Then, the player must decide whether it is an advantage to switch curtains or to insist on the original guess. At this point, we suggest the reader take a minute to ponder the problem and decide whether switching increases the chances of winning.

The decision the player faces is a tricky problem in elementary probability which has become a popular puzzle known as "the three curtains puzzle." It first appears to most people, including the authors of this article, that the player should be indifferent to which curtain is selected because both of the unopened curtains have the same probability of containing the prize. This position of indifference appeals to intuition in an apparently compelling manner. In Cecil Adams' words (see his "Straight Dope" column of November 23, 1990):

> If there are three curtains the chances of picking the right one are one in three. Knock one out of contention and the chances of either of the remaining curtains being the right one are even — one in two.

More careful examination of the conditional probabilities reveals that the indifference argument is wrong. The player's first guess contains the prize with probability 1/3 whereas the other curtain contains the prize with probability 2/3 (this fact is proved in the next section). Thus the player is not only better off by switching to the third curtain but, by doing so, doubles the chances of winning the prize.

The three curtains problem appeared first in a Letter to the Editor by Steve Selvin in *The American Statistician* in 1975. Recently, the problem received wide public attention after Marilyn vos Savant posed, and correctly answered, the puzzle in her "Ask Marilyn" column of September 9, 1990. Two more *Ask Marilyn* columns have followed (December 2, 1990 and February 17, 1991) in which Ph.D.'s from across the country claimed that Marilyn was wrong and that the indifference position was correct. The following excerpts are taken from those columns:

> May I suggest that you obtain and refer to a standard textbook in probability before you try to answer a question of this kind again.

If one curtain is shown to be a loser, that information changes the probability of either remaining choice to 1/2. As a professional mathematician, I'm very concerned with the general lack of mathematical skills. Please help by confessing your error.

You are utterly incorrect about the game-show question, and I hope this controversy will call some public attention to the serious national crisis in mathematical education.

Prompted by a reader who "stumbled upon" the "Ask Marilyn" column while "perversely flipping" through the *Parade* Magazine section of his Sunday newspaper, Cecil Adams took the wrong side in his first column on this puzzle (see the quotation above). Two columns followed: In the first he corrected his mistake, and in the second he dealt with letters from readers incredulous of seeing him abandon the position of indifference (and join sides with Marilyn).

This article discusses the three curtains problem and outlines some interesting variations. First, we attempt to dispel any doubt that if the player changes curtains the player is twice as likely to win the prize by staying with the original choice. Then we relate the problem to a version of the three prisoners' dilemma. Finally, we discuss entertaining variants.

Solution

There are many arguments for verifying that switching curtains offers an advantage. Perhaps the shortest, though not the most intuitive, argument is the following. Initially all three curtains are equally likely to contain the prize and the player has a 1/3 chance to get the prize on the first guess. Suppose that the player decides to stay with the first guess, after the announcer opens another curtain. Clearly, nothing changes about the curtain the player has already selected. In particular, the probability of getting the prize with that curtain is the same as before (1/3). But now there are only two candidate curtains and one of them contains the prize with probability 1/3. Therefore, the other curtain contains the prize with probability $1-\frac{1}{3}=\frac{2}{3}$. Deciding never to switch curtains is equivalent to not having the option to switch; hence the odds of winning by not switching are one in three.

A less elegant, but perhaps more convincing solution is to list all the possible outcomes of the game (this, in fact, is Selvin's original solution) [see Table 1]. Counting all outcomes shows that the probability of winning is $6/9 = 2/3$ when the player changes curtains.

Table 1

Prize is behind curtain	Player chooses curtain	Monty Hall opens curtain	Player switches	Player wins
1	1	2 or 3	from 1 to 2 or 3	No
1	2	3	from 2 to 1	Yes
1	3	2	from 3 to 1	Yes
2	1	3	from 1 to 2	Yes
2	2	1 or 3	from 2 to 1 or 3	No
2	3	1	from 3 to 2	Yes
3	1	2	from 1 to 3	Yes
3	2	1	from 2 to 3	Yes
3	3	1 or 2	from 3 to 1 or 2	No

The critical element behind the solution of the puzzle is that Monty knows *both* which curtain contains the prize and which curtain the player has chosen *before* he opens a curtain. He is deprived of the opportunity to eliminate the player's guess even if that curtain is empty. Therefore, Monty's choice does not contribute anything to the probability that the player's curtain contains the prize. On the other hand, the fact that, *having had the opportunity to do so*, Monty did not open the curtain which the player did not choose does increase the probability of finding the prize behind that curtain. This observation provides a

Three Prisoners' Dilemma

A more striking formulation of the same problem is one version of the well-known three prisoners' dilemma. There are three prisoners and one guard. One of the prisoners will be executed, and only the guard knows who that prisoner is. The night before the execution the guard brings food to the first prisoner. The prisoner is anxious about the upcoming execution and asks the guard whether he is the unlucky inmate. The guard replies that he is not allowed to tell the prisoner if it is he that will be executed, but he informs the prisoner that the third prisoner will be set free. The prisoner is greatly alarmed by this response because he immediately infers that the probability of his death has increased from 1/3 to 1/2.

In reality, however, the prisoner should not be any more concerned than he was before talking to the guard. His hasty conclusion is false, since he will still be executed with probability 1/3. Instead, it is the second prisoner that will be executed with higher probability. By analogy to the game with the curtains, the probability that the second prisoner will be executed is now raised to 2/3. This is the beauty of the apparent paradox.

heuristic explanation for why the curtain which the player did not choose is more likely to contain the prize.

Monty Hall himself is credited by Selvin with providing the following solution to the puzzle his show originated. Monty begins by confessing that he is not "a student of statistics problems," and by (wrongly) finding fault in Selvin's original argument. He goes on to add: "Oh, and incidentally, after one curtain is seen to be empty, the player's chances are no longer 50/50 but remain what they were in the first place, one out of three. It just seems to the contestant that one curtain having been eliminated, the player stands a better chance. Not so." Selvin remarks that it could not have been said better!

Monty's Behavior

Several assumptions on Monty Hall's behavior underlie the counterintuitive solution to the three curtains problem. One of them is that Monty knows which curtain contains the prize. If this were not the case, and all the remaining assumptions were the same as before, the indifference argument would be correct indeed: The player's chances of winning the game would be the same if the player switched curtains as if she did not. In both cases, the player would win one third of the time and lose one third of the time. The remaining third of the games Monty would accidentally open the curtain with the prize and the game would have to be played again.

Another major assumption is that there is nothing systematic in Monty's behavior which could help the player decide whether to switch curtains or not. If this assumption is dropped, the optimal strategy may change substantially. The following two scenarios, due to Timothy Chow from Princeton University, eloquently illustrate this possibility:

1. If the player picked the right curtain, Monty Hall is so impressed that he gives the player a million dollars and tells the player to go home, otherwise he lets the player reconsider. In this case switching is a sure bet.
2. If the player picked the wrong curtain, Monty is so angered that he shoots the player and the player dies, otherwise the show drags on. In this case, staying put is a sure bet.

These scenarios indicate that watching the show many times before deciding on an optimal strategy may prove useful. As an example, consider a situation suggested by Cecil Adams' second column. Suppose an attentive viewer of the show notes that Monty (a) does not give the player a chance to switch curtains on every show, and (b) decides to open a certain curtain twice as often when the player initially chooses the "right" curtain as does not. A player who is aware of these observations should remain indifferent between switching curtains and staying with the original choice: In both cases the chances of winning are exactly the same.

A Game with Many Rounds

Suppose that, motivated by all the attention he has recently received from "students of statistics problems," Monty Hall decides to become innovative and change the game as follows. There are 7 curtains (numbered from 1 to 7) only one of which contains the prize, and the game consists of three rounds. At the beginning of every round, Monty discards one of the curtains by opening one from the curtains that do not contain the prize and that differ from the player's current choice. Then, he gives the contestant the opportunity to switch to one of the remaining curtains. Under these conditions, a hypothetical game could develop as follows. The player chooses curtain 7 (the game begins); Monty discards curtain 3 and the player switches to curtain 5 (first round completed); Monty discards curtain 1 and the player moves to curtain 4 (second round completed); Monty discards curtain 7 and the player refuses Monty's offer to switch curtains (third round completed); Monty opens the curtain with the prize (the game ends). Suppose further that Monty wants to give the contestant a fair chance and decides to behave in a simple way. On every round, he chooses to open one of the available curtains (i.e., those that do not contain the prize and do not correspond to the player's current choice) *at random.*

In this game the decisions the player must make are considerably more complicated than in the original game, because the number of strategies at the player's disposal is much larger. The player must decide on every round whether to switch curtains or not; hence the player has access to at least $2^3 = 8$ possible strategies. In reality, there are many more additional choices to make. For example, once the player has decided to switch, the player must decide between moving to a curtain that has not been chosen before and moving to a curtain chosen on one (or even two) of the preceding rounds.

Which strategy should the contestant follow? Should the player switch on every round or only on the first round? Should the player switch on the first and last rounds? And when the player switches curtains, should the player prefer a curtain that has not been visited before or a curtain that has already been chosen on a previous round (assuming, of course, that such a curtain is available)? Again, it is probably worthwhile for the

Unconditional vs. Conditional Probabilities

As a generalization intended to recreate more faithfully the actual conditions on "Let's Make a Deal," suppose that the three curtains conceal a fabulous first prize (say a new car), a nice second prize (say a refrigerator), and a worthless gag third prize (say a goat). Suppose that Monty, in an effort to entice you, the player, to switch away from your original door selection, will always open the door revealing the refrigerator if he can (i.e., if you have not already chosen the refrigerator); otherwise, he will reveal the goat (he would never reveal the car's location). If this is Monty's strategy *and you know this*, then conditional on seeing the goat revealed, switching doors will of course win the car with probability 1 (Monty only does this if you initially picked the refrigerator). But conditional on seeing the refrigerator revealed, switching doors will win the car with probability only 1/2.

Note however that an argument similar to the one of Table 1 shows that *unconditionally*, the strategy of always switching doors still wins the car 2/3 of the time, as in the original problem.

reader to pause at this point and try to guess what the player's best strategy might be.

In an informal way, the optimal strategy can be derived from the intuition we gave for the solution to the original puzzle (a formal derivation is given in the Engel and Venetoulias paper in the Additional Reading). When Monty opens a curtain, he is prevented from discarding the player's current curtain and, therefore, is not giving out any information on the likelihood of finding the prize behind that curtain. By the same token, Monty's choice increases the likelihood of finding the prize behind one of the curtains he could, but did not, choose. Thus, on any given round, the curtains that are most likely to contain the prize are those that the player has not visited so far. The probability of finding the prize behind an "unvisited" curtain grows after each round, whereas the probability of finding the prize behind one of the remaining curtains (i.e., the player's current choice) remains unchanged. The last time the player has the option to switch curtains, the curtains that are more likely to contain the prize are those curtains that the player has never visited before. Furthermore, the chances of finding the prize behind one of those unvisited curtains do not depend on the strategy that the player followed during the earlier part of the game. The only effect the previous rounds have is to prevent Monty from discarding certain curtains and this is taken into account by all the *move-to-a-new-curtain-on-the-last-round* strategies. Choosing an unvisited curtain on the last round gives the contestant the largest probability of winning the prize regardless of what the player did on all the previous rounds. In other words, any strategy that switches to an unvisited curtain on the last round of the game is optimal.

Clearly, the optimal strategy depends neither on the actual number of curtains nor on the number of rounds, as long as there are enough curtains in the game. The probability of winning the prize in the general game with n curtains and k rounds can be calculated easily from the simplest among all optimal strategies. This strategy switches curtains *only* on the last round. With this strategy, the player always moves to a curtain not visited before, because the first, and only, time the player switches curtains is on the last round. The game then becomes equivalent to a game with one round where Monty discards k curtains and offers the player the option to switch to one of the remaining $n-k-1$ curtains. Counting cases, just as in Table 1, shows that the player's probability of winning is $(n-1)/[n(n-k-1)]$. For example, when there are 7 curtains and 3 rounds, the probability of winning is 2/7, which is twice as large as the probability of winning when the player stays with the initial choice throughout the game.

Conclusion

The surprising conclusion of the three curtains problem has made it a favorite mathematical puzzle. In fact, some prestigious business schools (e.g., Harvard, Stanford, etc.) use the puzzle to educate their MBA students in decision making. The problem serves as an example of how intuition can be misleading in making decisions under uncertainty (for more on this and other relevant entertaining problems, see Tversky and Gilovich's article, as well as the one by Larkey et al., in the Additional Reading).

Additional Reading

Adams, C., "The Straight Dope," syndicated column that appears in several weekly magazines; November 23, 1990; December 5, 1990; February 22, 1991.

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Selvin, S. (1975). A problem in probability, in "Letters to the Editor," *The American Statistician*, 29, 67 and 134.

Tversky, A., and Gilovich, T. (1989). "The cold facts about the 'hot hand' in basketball," *Chance*, Winter, 16–21.

vos Savant, M., "*Ask Marilyn*," syndicated column that appears in several weekly magazines; September 9, 1990; December 2, 1990; February 17, 1991.